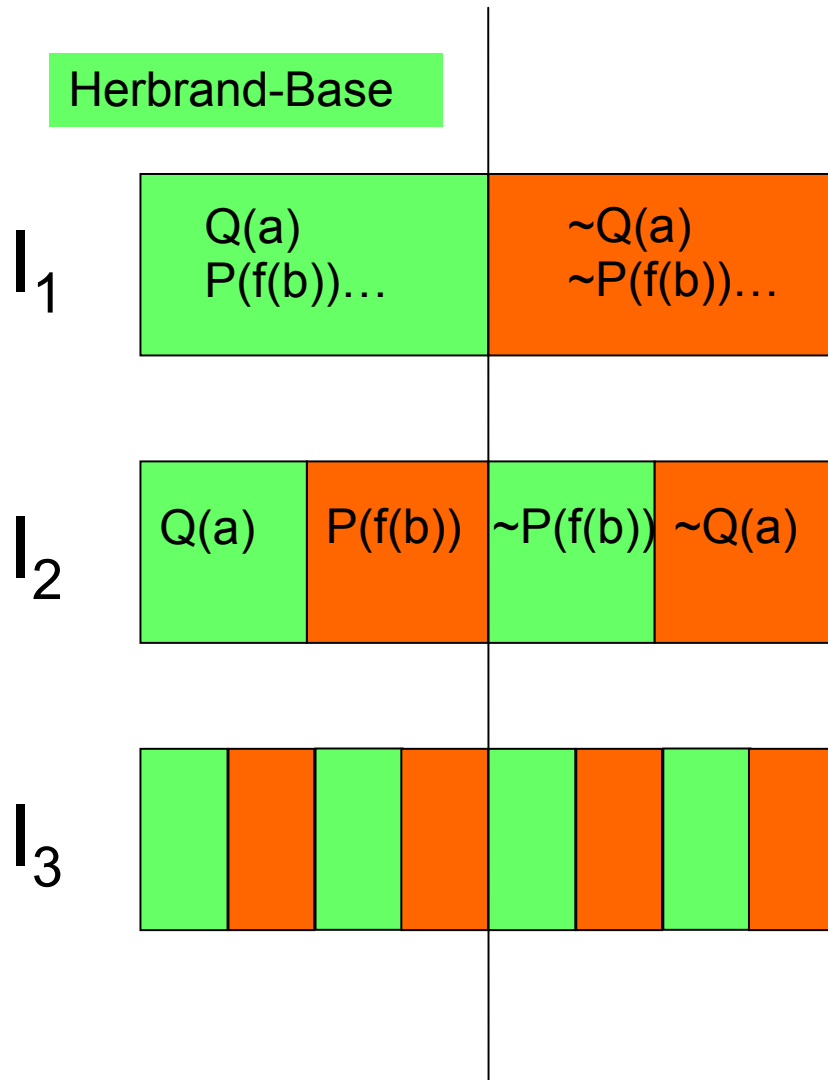


Definite Programs

Herbrand-Interpretations



The true atoms of the Herbrand base correlate with the corresponding interpretation.

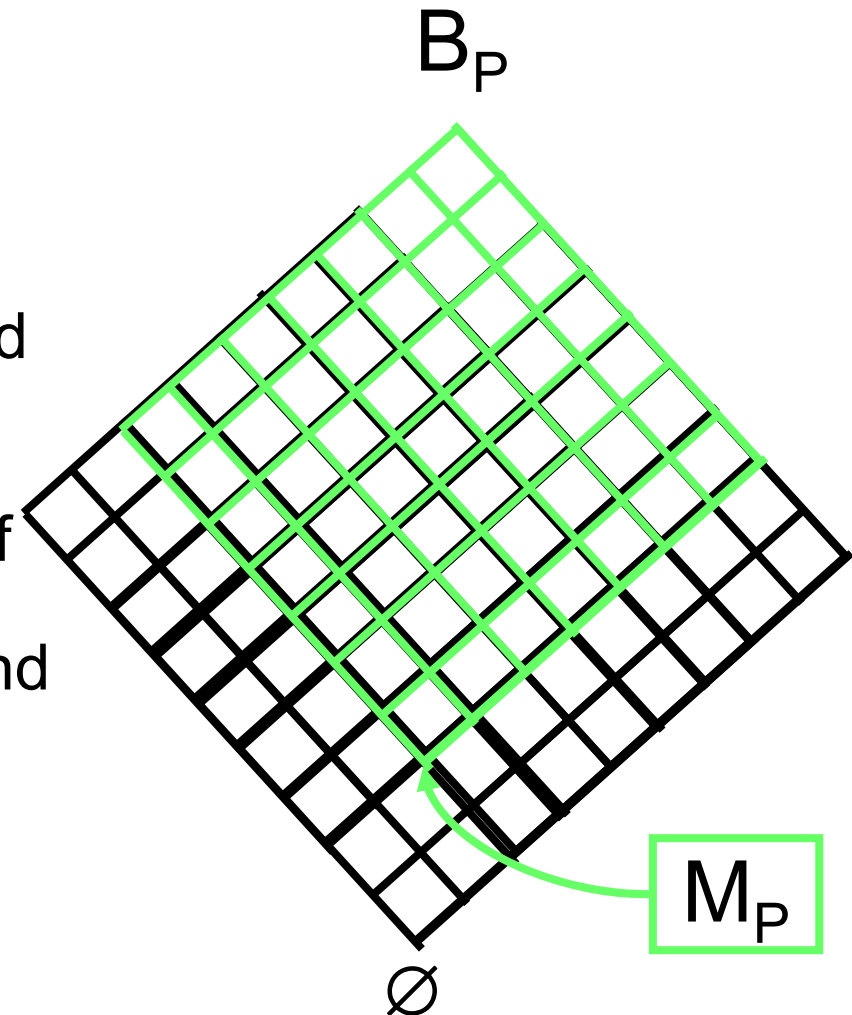


Proposition:

Let P be a definite program and $\{M_i\}_{i \in I}$ a non-empty set of Herbrand models for P . Then $\bigcap_{i \in I} M_i$ is an Herbrand model for P .
If $\{M_i\}_{i \in I}$ contains all Herbrand models for p , then $M_P := \bigcap_{i \in I} M_i$ is the *least Herbrand model* for P .

Idea

- $(2^{B_P}, \subseteq)$ is a lattice of all Herbrand interpretations of P with the bottom element \emptyset and the top element B_P
- The least upper bound (lub) of a set of interpretations is the union, the greatest lower bound is the intersection.



Proposition:

Let P be a definite program and $\{M_i\}_{i \in I}$ a non-empty set of Herbrand models for P . Then $\bigcap_{i \in I} M_i$ is an Herbrand model for P .
If $\{M_i\}_{i \in I}$ contains all Herbrand models for p , then
 $M_P := \bigcap_{i \in I} M_i$ is the *least Herbrand model* for P .

Proof:

$\bigcap_{i \in I} M_i$ is an Herbrand interpretation. Show, that it is a model.
Each definite program has B_P as model, hence I is not empty and one can show that M_P is a model.

Idea:

M_P is the „most natural“ model for P .

Theorem:

Let P be a definite program. Then
 $M_P = \{A \in B_P : A \text{ is a logical consequence of } P\}.$

Proof:

A is a logical consequence of P iff
 $P \cup \{\sim A\}$ is unsatisfiable iff
 $P \cup \{\sim A\}$ has no Herbrand model iff
 A is true wrt all Herbrand models of P iff
 $A \in M_P.$

Definition

Let L be a lattice and $T:L \rightarrow L$ a mapping.
 T is called *monotonic*, if $x \leq y$ implicates, that
 $T(x) \leq T(y)$.

Properties of Lattices

Definition

Let L be a lattice and $X \subseteq L$, X is called *directed*, if each finite sub-set of X has an upper bound in X .

Definition

Let L be a lattice and $T:L \rightarrow L$ a mapping.
 T is called *continuous*, if for each directed subset X
 $T(\text{lub}(X)) = \text{lub}(T(X))$.

Van Emden & Kowalski: The Semantics of Predicate Logic as a Programming Language,
J. ACM 23, 4, 1976, pp. 733-742.

Definition

Let P be a definite program. The mapping $T_P: 2^{B_P} \rightarrow 2^{B_P}$ is defined as follows: Let I be an Herbrand interpretation. Then

$$T_P(I) = \{ A \in B_P : A \leftarrow A_1 \wedge \dots \wedge A_n \\ \text{is a ground instance of a clause in } P \text{ and} \\ \{A_1, \dots, A_n\} \subseteq I \}$$

Practice

Let P be a definite program.

$\text{even}(f(f(x))) \leftarrow \text{even}(x).$
 $\text{odd}(f(x)) \leftarrow \text{even}(x).$
 $\text{even}(0).$

Let $I_0 = \emptyset.$

Then

$I_1 = T_P(I_0) = ?$

$T_P(I) = \{A \in B_P : A \leftarrow A_1 \wedge \dots \wedge A_n$
is a ground instance of a
clause in P and
 $\{A_1, \dots, A_n\} \subseteq I\}$

Example

Let P be a definite program.

$\text{even}(f(f(x))) \leftarrow \text{even}(x).$
 $\text{even}(0).$

Let $I_0 = \emptyset$.

Then

$I_1 = T_P(I_0) = \{\text{even}(0)\},$

$I_2 = T_P(I_1) = \{\text{even}(0), \text{even}(f(f(0)))\},$

$I_3 = T_P(I_2) = \{\text{even}(0), \text{even}(f(f(0))),$
 $\text{even}(f(f(f(f(0))))\}, \dots$

$T_P(I) = \{A \in B_P : A \leftarrow A_1 \wedge \dots \wedge A_n$
is a ground instance of a
clause in P and
 $\{A_1, \dots, A_n\} \subseteq I\}$

T_P is monotonic.

Let P be a definite program.

$\text{plus}(x, f(y), f(z)) \leftarrow \text{plus}(x, y, z)$
 $\text{plus}(f(x), y, f(z)) \leftarrow \text{plus}(x, y, z)$
 $\text{plus}(0, 0, 0)$

Let $I_0 = \emptyset$.

Then

$I_1 = T_P(I_0) = ?$

$T_P(I) = \{A \in B_P : A \leftarrow A_1 \wedge \dots \wedge A_n$
is a ground instance of a
clause in P and
 $\{A_1, \dots, A_n\} \subseteq I\}$

Let P be a definite program.

$\text{plus}(x, f(y), f(z)) \leftarrow \text{plus}(x, y, z)$
 $\text{plus}(f(x), y, f(z)) \leftarrow \text{plus}(x, y, z)$
 $\text{plus}(0, 0, 0)$

$T_P(I) = \{A \in B_P : A \leftarrow A_1 \wedge \dots \wedge A_n$
 is a ground instance of a
 clause in P and
 $\{A_1, \dots, A_n\} \subseteq I\}$

Let $I_0 = \emptyset$.

Then

$I_1 = T_P(I_0) = \{\text{plus}(0, 0, 0)\}$

$I_2 = T_P(I_1) = \{\text{plus}(0, 0, 0),$
 $\text{plus}(f(0), 0, f(0)),$
 $\text{plus}(0, f(0), f(0))\}$

$I_3 = T_P(I_2) = \{\text{plus}(0, 0, 0),$
 $\text{plus}(f(0), 0, f(0)),$
 $\text{plus}(0, f(0), f(0)),$
 $\text{plus}(f(f(0)), 0, f(f(0))),$
 $\text{plus}(0, f(f(0)), f(f(0))),$
 $\text{plus}(f(0), f(0), f(f(0))),$
 $\text{plus}(f(0), f(0), f(f(0)))\} \dots$

Proposition

Let P be a definite program and I be an Herbrand interpretation of P . Then I is a model for P iff $T_P(I) \subseteq I$.

$\Rightarrow \checkmark$

Let I be an Herbrand interpretation, which is not a model of P and $T_P(I) \subseteq I$. Then there exist ground instances $\{\sim A, A_1, \dots, A_n\} \in I$ and a clause $A \leftarrow A_1 \wedge \dots \wedge A_n$ in P . Then $A \in T_P(I) \subseteq I$. Refutation. ⚡

Definition

let L be a complete lattice and $T:L \rightarrow L$ be monotonic. Then we define

$$T \uparrow 0 = \perp$$

$$T \uparrow \alpha = T(T \uparrow (\alpha - 1)), \text{ if } \alpha \text{ is a successor ordinal}$$

$$T \uparrow \alpha = \text{lub}\{T \uparrow \beta : \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}$$

$$T \downarrow 0 = T$$

$$T \downarrow \alpha = T(T \downarrow (\alpha - 1)), \text{ if } \alpha \text{ is a successor ordinal}$$

$$T \downarrow \alpha = \text{lub}\{T \downarrow \beta : \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal}$$

Example

Theorem

Let P be a definite program.

Then $M_P = \text{lfp}(T_P) = T_P \uparrow \omega$, where

$T \uparrow \alpha = T(T \uparrow (\alpha - 1))$, if α is a successor ordinal

$T \uparrow \alpha = \text{lub}\{T \uparrow \beta : \beta < \alpha\}$, if α is a limit ordinal

Proof

$M_P = \text{glb}\{I : I \text{ is an Herbrand model for } P\}$

$= \text{glb}\{I : T_P(I) \subseteq I\}$

$= \text{lfp}(T_P)$

(not shown here)

$= T_P \uparrow \omega$.

Definition

Let P be a definite program and G a definite goal. An *answer* for $P \cup \{G\}$ is a substitution for variables of G .

Definition

Let P be a definite program, G a definite goal $\leftarrow A_1 \wedge \dots \wedge A_n$ and θ an answer for $P \cup \{G\}$.

We say that θ is a *correct answer* for $P \cup \{G\}$, if $\forall ((A_1 \wedge \dots \wedge A_n)\theta)$ is a logical consequence of P .
The answer „no“ is *correct* if $P \cup \{G\}$ is satisfiable.

Example

Theorem

Let P be a definite program and G a definite goal

$\leftarrow A_1 \wedge \dots \wedge A_n$. Suppose θ is an answer for $P \cup \{G\}$, such that $(A_1 \wedge \dots \wedge A_n)\theta$ is ground. Then the following are equivalent:

1. θ is correct
2. $(A_1 \wedge \dots \wedge A_n)\theta$ is true wrt every Herbrand model of P
3. $(A_1 \wedge \dots \wedge A_n)\theta$ is true wrt the least Herbrand model of P .

Proof

it suffices to show that 3 implies 1

$(A_1 \wedge \dots \wedge A_n)\theta$ is true wrt the least Herbrand model of P implies

$(A_1 \wedge \dots \wedge A_n)\theta$ is true wrt all Herbrand models of P implies

$\sim(A_1 \wedge \dots \wedge A_n)\theta$ is false wrt all Herbrand models of P implies

$P \cup \{\sim(A_1 \wedge \dots \wedge A_n)\theta\}$ has no Herbrand models

implies $P \cup \{\sim(A_1 \wedge \dots \wedge A_n)\theta\}$ has no models.